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DIFFERENTIAL EQUATIONS

1. INTRODUCTION

An equation containing an independent variable, dependent variable and differential coefficients is called a differential equation.

(i) $\frac{dy}{dx} = \sin x$ (ii) $\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^3 = 0$ (iii) $\left(\frac{d^4y}{dx^4}\right)^3 - 4\frac{dy}{dx} = 5\cos 3x$

2. ORDER OF DIFFERENTIAL EQUATION

The order of a differential equation is the order of the highest derivative occurring in the differential equation. For example, the order of the above mentioned differential equations are 1, 2, and 4 respectively.

3. DEGREE OF DIFFERENTIAL EQUATION

The degree of a differential equation is the degree of the highest order derivative when differential coefficients are free from radicals and fractions. For example the degrees of above differential equations are 1, 2, and 3 respectively.

Differential Equation	Order of D.E.	Degree of D.E
$\frac{dy}{dx} + 4y = \sin x$	1	1
$\left(\frac{d^2y}{dx^2}\right)^4 + \left(\frac{dy}{dx}\right)^5 - y = e^x$	2	4
$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3y = \cos x$	2	1
$\frac{dy}{dx} = \frac{x^4 - y^4}{xy\left(x^2 + y^2\right)}$	1	1

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Differential Equation	Order of D.E.	Degree of D.E
$y = x\frac{dy}{dx} + \sqrt{a^2\left(\frac{dy}{dx}\right)^2 + b^2}$ $\Rightarrow (x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + (y^2 - b^2) = 0$	1	2
$\frac{d^2y}{dx^2} = \left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2} \Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 - \left(1+\left(\frac{dy}{dx}\right)^2\right)^3 = 0$	2	2

4. CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Differential equations are first classified according to their order. First-order differential equations are those in which only the first order derivative, and no higher order derivatives appear. Differential equations of order two or more are referred to as higher order differential equations.

A differential equation is said to be linear if the unknown function, together with all of its derivatives, appears in the differential equations with a power not greater than one and not as products either. A nonlinear differential equation is a differential equation which is not linear.

e.g. $y' + y = 0$ is a linear differential equation,

$y'' + yy' + y^2 = 0$ is a non linear differential equation,

Procedure to form a differential equation that represents a given family of curves

Case I:

If the given family F1 of curves depends on only one parameter then it is represented by an equation of the form $F1(x, y, a) = 0$ ... (i)

For example, the family of parabolas $y^2 = ax$ can be represented by an equation of the form

$f(x, y, a): y^2 = ax$

Differentiating equation (i) with respect to x, we get an equation involving y′, y, x and a.

$g(x, y, y', a) = 0$ ... (ii)

The required differential equation is then obtained by eliminating a from equation (i) and (ii) as

$F(x, y, y') = 0$ ... (iii)

Case II:

If the given family F2 of curves depends on the parameters a, b (say) then it is represented by an equation of the form $F2(x, y, a, b) = 0$ ... (iv)

Differentiating equation (iv) with respect to x, we get an equation involving y′, x, y, a, b.

$g(x, y, y', a, b) = 0$ ... (v)

Now we need another equation to eliminate both a and b. This equation is obtained by differentiating equation (v), wrt x, to obtain a relation of the form $h(x, y, y', y'', a, b) = 0$ ... (vi)

The required differential equation is then obtained by elimination a and b from equations (iv), (v) and (vi) as $F(x, y, y', y'') = 0$ ... (vii)

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Note: The order of a differential equation representing a family of curves is the same as the number of arbitrary constants present in the equation corresponding to the family of curves.

5. FORMATION OF DIFFERENTIAL EQUATIONS

If an equation is dependent and dependent variables having some arbitrary constant are given, then the differential equation is obtained as follows:

(a) Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.

(b) Eliminate the arbitrary constants.

(c) Hence on eliminating arbitrary constants results a differential equation which involves x, y, $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ .... $\frac{d^my}{d_xm}$ (where m=number of arbitrary constants).

Illustration 1: Form the differential equation corresponding to $y^2 = m(a^2 - x^2)$, where m and a are arbitrary constants.

Sol: Since the given equation contains two arbitrary constant, we shall differentiate it two times with respect to x and we get a differential equation of second order.

We are given that $y^2 = m(a^2 - x^2)$ ... (i)

Differentiating both sides of (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = m(-2x) \Rightarrow y\frac{dy}{dx} = -mx \qquad \text{... (ii)}$$

Differentiating both sides of (ii) w.r.t. to x, we get $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = -m$ ... (iii)

From (ii) and (iii), we get, $x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right] = y\frac{dy}{dx}$

This is the required differential equation.

Illustration 2: Form diff. equation of $ax^2 + by^2 = 1$

Sol: Similar to the above problem the given equation contains two arbitrary constants, so we shall differentiate it two times with respect to x and then by eliminating a and b we get the differential equation of second order.

$$ax^2 + by^2 = 1 \Rightarrow \quad 2ax + 2by\frac{dy}{dx} = 0 \Rightarrow \quad a + b\,(yy'' + (y')^2) = 0$$

Eliminating a and b we get $\frac{y}{x}y' = yy'' + (y')^2 \quad \Rightarrow \quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - \frac{y}{x}\frac{dy}{dx} = 0$

Illustration 3: Form the differential equation corresponding to $y^2 = a(b^2 - x^2)$, where a and b are arbitrary constants.

Sol: Similar to illustration 1.

We have, $y^2 = a(b^2 - x^2)$ ... (i)

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In this equation, there are two arbitrary constants a, b, so we have to differentiate twice, Differentiating the given equation (i) w.r.t. 'x'. We get $2y\frac{dy}{dx} = -2x.a \Rightarrow y\frac{dy}{dx} = -ax$ ... (ii)

Differentiating (ii) with respect to x, we get $y\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{dy}{dx} = -a \Rightarrow y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -a$ ... (iii)

Substituting the value of a in (ii), we get

$$y\frac{dy}{dx} = \left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)2\right\}x \Rightarrow y\frac{dy}{dx} = xy\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \Rightarrow xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

Illustration 4: Find the differential equation of the following family of curves: $xy = Ae^x + Be^{-x} + x^2$

Sol: Here in this problem A and B are the two arbitrary constants, hence we shall differentiate it two times with respect to x and then by eliminating constant terms we will get the required differential equation.

Given: $xy = Ae^x + Be^{-x} + x^2$ ... (i)

Differentiating (i) with respect to 'x', we get $x\frac{dy}{dx} + y = Ae^x - Be^{-x} + 2x$

Again differentiating with respect to 'x', we get

$$x\frac{d^2y}{dx^2} + 1\frac{dy}{dx} + 1.\frac{dy}{dx} = Ae^x + Be^{-x} + 2 \Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy - x^2 + 2$$

Illustration 5: Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is a general solution of the differential equation $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$

Sol: Here only one arbitrary constant is present hence we shall differentiate it one time with respect to x and then by substituting the value of c we shall prove the given equation.

Let us find the differential equation for $x^2 - y^2 = c(x^2 + y^2)^2$ ... (i)

Differentiating (i), with respect to 'x', we get $2x - 2y\frac{dy}{dx} = c.2\left(x^2 + y^2\right)\left(2x + 2y\frac{dy}{dx}\right)$ ... (ii)

Substituting the value of c from (i) in (ii), we get

$$\Rightarrow x - y\frac{dy}{dx} = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}\left(x^2 + y^2\right)\left(2x + 2y\frac{dy}{dx}\right) \Rightarrow (x^2 + y^2)\left(x - y\frac{dy}{dx}\right) = (x^2 - y^2)\left(2x + 2y\frac{dy}{dx}\right)$$

$$\Rightarrow [2y(x^2 - y^2) + y(x^2 + y^2)]\frac{dy}{dx} = x(x^2 + y^2) - 2x(x^2 - y^2) \Rightarrow (3x^2y - y^3)\frac{dy}{dx} = 3xy^2 - x^3$$

$\Rightarrow (x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$ As this equation matches the one given in the problem statement. Hence the given equation is the solution for the differential equation.

Hence proved.

Illustration 6: Find the differential equation of the family of curves $y = e^x(a\cos x + b\sin x)$

Sol: Since given family of curves have two constants a and b, so we have to differentiate twice with respect to x.

We have, $y = e^x(a\cos x + b\sin x)$ ... (i)

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Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = e^x(a\cos x + b\sin x) + e^x(-a\sin x + b\cos x) = y + e^x(-a\sin x + b\cos x)$$

$$\Rightarrow \quad \frac{dy}{dx} - y = e^x(-a\sin x + b\cos x) \qquad \text{... (ii)}$$

Differentiating (ii) with respect to x, we get

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x(-a\sin x + b\cos x) + e^x(-a\cos x - b\sin x) = \frac{dy}{dx} - y - e^x(a\cos x + b\sin x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} = \frac{dy}{dx} - y - y \qquad [\because e^x(a\cos x + b\sin x) = y] \Rightarrow \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

This is the required differential equation.

Illustration 7: Find the differential equation of all circles which pass through the origin and whose centers lie on the y axis.

Sol: As circles passes through the origin and whose centers lie on the y axis hence g = 0 and point (0, 0) will satisfy general equation of given circle.

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (i)}$$

Since it passes through origin (0, 0), it will satisfy equation (i)

$$\Rightarrow \quad (0)^2 + (0)^2 + 2g.(0) + 2f.(0) + c = 0 \quad \Rightarrow \quad c = 0$$

$$\Rightarrow \quad x^2 + y^2 + 2gx + 2fy = 0$$

This is the equation of a circle with center (–g, –f) and passing through the origin.

If the center lies on the y-axis, we have g = 0,

$$\Rightarrow \quad x^2 + y^2 + 2.(0).x + 2fy = 0 \qquad \Rightarrow \quad x^2 + y^2 + 2fy = 0 \qquad \text{... (ii)}$$

Hence, (ii) represents the required family of circles with center on y axis and passing through origin.

Differentiating (ii) with respect to x, we get

$$2x + 2y\frac{dy}{dx} + 2f\frac{dy}{dx} = 0 \ \Rightarrow f = -\left\{\frac{x + y.\left(\frac{dy}{dx}\right)}{\left(\frac{dy}{dx}\right)}\right\}$$

Substituting this value of f in (2), we get

$$x^2 + y^2 - 2y\left(\frac{x + y.\left(\frac{dy}{dx}\right)}{\left(\frac{dy}{dx}\right)}\right) = 0 \Rightarrow (x^2 + y^2)\frac{dy}{dx} - 2xy - 2y^2\left(\frac{dy}{dx}\right) = 0 \Rightarrow (x^2 - y^2)\frac{dy}{dx} - 2xy = 0$$

This is the required differential equation.

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